LECTURE NOTES ON DUALITY AND INTERPOLATION SPACES

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Abstract. Known or essentially known results about duals of interpolation spaces are presented, taking a point of view sometimes slightly different from the usual one. Particular emphasis is placed on Alberto Calderón's theorem describing the duals of complex interpolation spaces. The pace is slow, since these notes are intended for graduate students who have just begun to study interpolation spaces.

This paper, or set of lecture notes, gives a careful, pedantic, perhaps even a bit too pedantic, treatment of the general topic of duality in the context of Banach couples. I will follow the same standard (it might be called "naive") approach which is explicitly or implicitly used in many papers about interpolation spaces, including the classical expositions of Alberto Calderón [3] and Lions–Peetre [13]. It should be mentioned that several authors, notably Sten Kaijser and Joan Wick–Pelletier (see [9], [10], [11] and [16]) and subsequently Yuri Brudnyi and Natan Krugljak (see [2] pp. 268–282) have sometimes found it preferable to use a different and more sophisticated approach, involving so–called "dolittle diagrams", to deal with duality for Banach couples.

We will slowly work our way through many well known facts. In particular I will give a rather detailed proof of Calderón's famous duality theorem for his complex interpolation spaces, using an approach with some differences from his original proof. I believe that in some ways, despite its considerable length, my proof of that theorem is simpler than the original one, and draws attention to some hopefully useful details about Calderón's interpolation spaces, which have not been presented quite so explicitly in previous papers.

We will not make any explicit mention or use here of the remarkable alternative characterizations, due to Svante Janson [8], of the complex method of interpolation via weighted sequence spaces of Fourier coefficients. These characterizations provide an alternative way of describing the dual spaces of Calderón's spaces. (Cf. [8] Theorems 1 and 2 on pp. 53–55 and, in particular, Theorem 23 on p. 69.)

I apologize to those readers who might find many of the details mentioned here to be superflous. But one of my aims is to make these notes comfortably accessible to beginning graduate students, including those who might perhaps be looking at my recent paper [6] about complex interpolation of compact operators mapping into a couple of Banach lattices.

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1. Notation, basic definitions and properties of the dual couple of a Banach couple

All Banach spaces considered here will be assumed to be complex Banach spaces.

For any two Banach spaces, A and B it will be convenient to use the notation $A \overset{1}{\subset} B$ to mean that A is contained in B and that, furthermore, $\|a\|_B \le \|a\|_A$ for each $a \in A$. The notation $A \overset{1}{=} B$ will mean that $A \overset{1}{\subset} B$ and also $B \overset{1}{\subset} A$, i.e., it means that A and B coincide with equality of norms.

As usual, A^* will denote the space of all continuous linear functionals on A. But we will sometimes use the alternative notation $A^\#$ for another space which is isometrically isomorphic to A^*.

A ***Banach couple***, sometimes also referred to as a ***Banach pair***, and denoted by (X_0, X_1), is an ordered pair of Banach spaces X_0 and X_1 which are both contained in some Hausdorff topological vector space $\mathcal{X}$. The two inclusion maps $X_0 \subset \mathcal{X}$ and $X_1 \subset \mathcal{X}$ are both required to be continuous.

For each Banach couple (X_0, X_1), it is well known and immediately evident that the spaces $X_0 \cap X_1$ and $X_0 + X_1$ normed, respectively, by $\|x\|_{X_0 \cap X_1} = \max\left\{\|x\|_{X_0}, \|x\|_{X_1}\right\}$ and $\|x\|_{X_0+X_1} = \inf\left\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1\right\}$ are also Banach spaces.

Definition 1. *Let (X_0, X_1) be a Banach couple. We say that (X_0, X_1) is **regular** if $X_0 \cap X_1$ is dense in X_j for $j = 0, 1$. A Banach space X which satisfies $X_0 \cap X_1 \subset X \subset X_0 + X_1$ where both the inclusions are continuous, is said to be an **intermediate space** with respect to the couple (X_0, X_1). If, in addition, $X_0 \cap X_1$ is dense in X then we say that X is a **regular intermediate space** with respect to (X_0, X_1).*

We shall assume throughout these notes that (X_0, X_1) is a regular Banach couple of complex Banach spaces.

The notation $\langle \cdot, \cdot \rangle$ will always denote the bilinear functional defined on $(X_0 \cap X_1) \times (X_0 \cap X_1)^*$. Thus, whenever we write $\langle x, y \rangle$, we will always be assuming that $x \in X_0 \cap X_1$ and $y \in (X_0 \cap X_1)^*$ and $\langle x, y \rangle$ is the value of the linear functional y acting on the element x. Later we shall define an explicit variant of this notation for the case where x is in a space larger than $X_0 \cap X_1$ and y is in a suitable space smaller than $(X_0 \cap X_1)^*$.

Definition 2. *Suppose that X is a regular intermediate space with respect to the couple (X_0, X_1). Let $X^\#$ denote the subspace of elements $y \in (X_0 \cap X_1)^*$ which satisfy*

$$\|y\|_{X^\#} := \sup\left\{|\langle x, y \rangle| : x \in X_0 \cap X_1, \|x\|_X \le 1\right\} < \infty.$$

Note in particular that $(X_0 \cap X_1)^\# \overset{1}{=} (X_0 \cap X_1)^*$.

Fact 3. *Suppose that U and V are both regular intermediate spaces with respect to (X_0, X_1) and $U \overset{1}{\subset} V$. Then $V^\# \overset{1}{\subset} U^\#$. More generally, if U is continuously embedded in V then $V^\#$ is continuously embedded in $U^\#$.*

Proof of Fact 3. This will follow immediately from the definition. We have $\|x\|_V \le c\|x\|_U$ for all $x \in X_0 \cap X_1$ and some constant c (and in particular we will

also consider the case where $c = 1$). Then $|\langle x, y\rangle| \le \|x\|_V \|y\|_{V^\#} \le c\|x\|_U \|y\|_{V^\#}$ for all $x \in X_0 \cap X_1$ and all $y \in V^\#$. Consequently $y \in U^\#$ with $\|y\|_{U^\#} \le c\|y\|_{V^\#}$. $\square$

In particular, Fact 3 tells us that, for each intermediate space X in which $X_0 \cap X_1$ is dense, we have that

(1) $\quad X^\# \subset (X_0 \cap X_1)^\# = (X_0 \cap X_1)^*$ and this inclusion is continuous.

For $j = 0, 1$ we shall denote $X_j^\# = (X_j)^\#$. Thus $X_0^\# \cap X_1^\#$ is the set of all elements $y \in (X_0 \cap X_1)^*$ which satisfy

$$\|y\|_{X_0^\# \cap X_1^\#} = \max\left\{\|y\|_{X_0^\#}, \|y\|_{X_1^\#}\right\} < \infty.$$

Since $X_0 \cap X_1$ is dense in $X_0 + X_1$, the space $(X_0 + X_1)^\#$ can also be defined as in Definition 2 and we claim that

Fact 4. *$X_0^\# \cap X_1^\#$ coincides isometrically with $(X_0 + X_1)^\#$.*

Proof. First, for each $x \in X_0 \cap X_1$ and each $y \in X_0^\# \cap X_1^\#$ and each decomposition of x in the form $x = x_0 + x_1$, where $x_j \in X_j$ (and so in fact $x_j \in X_0 \cap X_1$) for $j = 0, 1$, we have

$$\begin{aligned} |\langle x, y\rangle| &= |\langle x_0, y\rangle + \langle x_1, y\rangle| \le |\langle x_0, y\rangle| + |\langle x_1, y\rangle| \\ &\le \|x_0\|_{X_0} \|y\|_{X_0^\#} + \|x_1\|_{X_1} \|y\|_{X_1^\#} \\ &\le \left(\|x_0\|_{X_0} + \|x_1\|_{X_1}\right) \max\left\{\|y\|_{X_0^\#}, \|y\|_{X_1^\#}\right\} \\ &= \left(\|x_0\|_{X_0} + \|x_1\|_{X_1}\right) \|y\|_{X_0^\# \cap X_1^\#}. \end{aligned}$$

Taking the infimum over all decompositions $x = x_0 + x_1$ of the kind specified above, we obtain that

$$|\langle x, y\rangle| \le \|x\|_{X_0 + X_1} \|y\|_{X_0^\# \cap X_1^\#}.$$

Then, keeping y fixed and taking the supremum over all $x \in X_0 \cap X_1$ with $\|x\|_{X_0 + X_1} \le 1$, we obtain that $X_0^\# \cap X_1^\# \overset{1}{\subset} (X_0 + X_1)^\#$.

Conversely, if $y \in (X_0 + X_1)^\#$, then, for $j = 0, 1$ we have, for all $x \in X_0 \cap X_1$, that

$$|\langle x, y\rangle| \le \|x\|_{X_0 + X_1} \|y\|_{(X_0 + X_1)^\#} \le \|x\|_{X_j} \|y\|_{(X_0 + X_1)^\#}.$$

Taking the supremum over all $x \in X_0 \cap X_1$ with $\|x\|_{X_j} \le 1$, we obtain that $y \in X_j^\#$ with $\|y\|_{X_j^\#} \le \|y\|_{(X_0 + X_1)^\#}$. It follows that $(X_0 + X_1)^\# \overset{1}{\subset} X_0^\# \cap X_1^\#$. So indeed we have

$$(X_0 + X_1)^\# \overset{1}{=} X_0^\# \cap X_1^\# \tag{2}$$

and the proof of Fact 4 is complete. $\square$

It seems intuitively obvious that, for each regular intermediate space X, the space $X^\#$ can be identified with the dual space X^*. But let us now explain this more precisely:

Definition 5. *Suppose that X is a regular intermediate space with respect to the regular Banach couple (X_0, X_1).*

(i) For each $y \in X^\#$ and each $x \in X$ we define

$$\langle x, y\rangle_X := \lim_{n \to \infty} \langle x_n, y\rangle$$

where $\{x_n\}_{n\in\mathbb{N}}$ *is a sequence (any sequence) of elements in* $X_0 \cap X_1$ *which converges to* x *in* X *norm.*

(ii) In contexts where it can help avoid ambiguity, we will use the alternative and consistent notation $\langle\cdot,\cdot\rangle_{X_0\cap X_1}$ *for the bilinear functional* $\langle\cdot,\cdot\rangle$ *introduced at the beginning of this discussion.*

In many papers, notation such as $\langle x,y\rangle$, originally introduced for the case when $x \in X_0 \cap X_1$, is still employed instead of $\langle x,y\rangle_X$, even when x is not in $X_0 \cap X_1$, and apparently this usually does not cause any problems or ambiguities. However here we feel the need to be more explicit about the exact kind of duality being used at each step of our development.

Of course the value of $\langle x,y\rangle_X$ is independent of the particular choice of the sequence $\{x_n\}_{n\in\mathbb{N}}$ and of course we also have

$$|\langle x,y\rangle_X| \le \|x\|_X \|y\|_{X^\#} \text{ for each } x \in X \text{ and each } y \in X^\#. \tag{3}$$

Furthermore,

$$\begin{aligned} \|y\|_{X^\#} &= \sup\{|\langle x,y\rangle| : x \in X_0 \cap X_1, \|x\|_X \le 1\} \\ &= \sup\{|\langle x,y\rangle_X| : x \in X_0 \cap X_1, \|x\|_X \le 1\} \\ &\le \sup\{|\langle x,y\rangle_X| : x \in X, \|x\|_X \le 1\} \le \|y\|_{X^\#} , \end{aligned} \tag{4}$$

and in fact all the inequalities in (4) are equalities.

Let us use the notation $\lambda(x)$ for the action of a bounded linear functional $\lambda \in X^*$ on an element $x \in X$. We define the map $I_X : X^\# \to X^*$ by

$$(I_X y)(x) = \langle x,y\rangle_X \text{ for each } y \in X^\# \text{ and each } x \in X.$$

It is clear from (4) that $\|I_X y\|_{X^*} = \|y\|_{X^\#}$ for each $y \in X^\#$.

Now let λ be an arbitrary element of X^*. Since X is an intermediate space, for each $x \in X_0 \cap X_1$ we have $|\lambda(x)| \le \|\lambda\|_{X^*} \|x\|_X \le c\|\lambda\|_{X^*} \|x\|_{X_0\cap X_1}$ for some constant c. In other words, the restriction $\lambda|_{X_0\cap X_1}$ of λ to $X_0 \cap X_1$ is a bounded linear functional on $X_0 \cap X_1$ and thus an element of $(X_0\cap X_1)^* = (X_0 \cap X_1)^\#$. We can thus use the notation

$$\left\langle x, \lambda|_{X_0\cap X_1}\right\rangle = \lambda(x) \text{ for all } x \in X_0 \cap X_1. \tag{5}$$

Furthermore, since $X_0 \cap X_1$ is dense in X, the functionals $\lambda|_{X_0\cap X_1}$ and λ are in one to one correspondence, i.e., for two elements λ and σ of X^*, we have $\lambda|_{X_0\cap X_1} = \sigma|_{X_0\cap X_1}$ if and only if $\lambda = \sigma$. We deduce that the map $J_X : X^* \to (X_0 \cap X_1)^*$ defined by $J_X\lambda = \lambda|_{X_0\cap X_1}$ and (5) for each $\lambda \in X^*$ is one to one. Furthermore,

$$|\langle x, J_X\lambda\rangle| = |\lambda(x)| \le \|\lambda\|_{X^*} \|x\|_X \text{ for all } x \in X_0 \cap X_1,$$

and so, for each $\lambda \in X^*$, we have $J_X\lambda \in X^\#$ with $\|J_X\lambda\|_{X^\#} \le \|\lambda\|_{X^*}$, i.e., $J_X : X^* \to X^\#$ with norm not exceeding 1. It is clear that the operators J_X and I_X set up a one to one isometry between X^* and $X^\#$. Thus we have shown that

Fact 6. $X^\#$ *is exactly the dual space of* X*, if we adopt the convention that the bilinear functional* $\langle x,y\rangle_X$ *defined in Definition 5 specifies the action of each element* y *of the dual space on each element* x *of* X*.*

Now we can consider the couple consisting of the dual spaces $X_0^\#$ and $X_1^\#$.

Fact 7. *Let (X_0, X_1) be a regular Banach couple. Then $(X_0^\#, X_1^\#)$ is a Banach couple, and $(X_0 \cap X_1)^\# \overset{1}{=} X_0^\# + X_1^\#$.*

Proof. First we observe (cf. (1)) that $X_0^\#$ and $X_1^\#$ are both continuously embedded in $(X_0 \cap X_1)^*$. So $(X_0^\#, X_1^\#)$ is indeed a Banach couple. More precisely, since $X_0 \cap X_1 \overset{1}{\subset} X_j$, we have $X_j^\# \overset{1}{\subset} (X_0 \cap X_1)^*$ for $j = 0, 1$ and so $X_0^\# + X_1^\# \overset{1}{\subset} (X_0 \cap X_1)^*$. As already mentioned, we have $(X_0 \cap X_1)^\# \overset{1}{=} (X_0 \cap X_1)^*$. It remains to show that

$$(X_0 \cap X_1)^\# \overset{1}{\subset} X_0^\# + X_1^\#. \tag{6}$$

Consider the Banach space $X_0 \oplus_\infty X_1$ which is the cartesian product of X_0 and X_1 normed by $\|x_0 \oplus x_1\|_{X_0 \oplus_\infty X_1} := \max\left\{\|x_0\|_{X_0}, \|x_1\|_{X_1}\right\}$. Adopting the conventions which were fixed above in Definition 5 and the subsequent discussion for realizing the dual space of each regular intermediate space with respect to the couple (X_0, X_1), we see that dual of $X_0 \oplus_\infty X_1$ is of course the space $X_0^\# \oplus_1 X_1^\#$, i.e., the cartesian product of $X_0^\#$ and $X_1^\#$ with norm $\|y_0 \oplus y_1\|_{X_0^\# \oplus_1 X_1^\#} = \|y_0\|_{X_0^\#} + \|y_1\|_{X_1^\#}$, and the action of linear functionals is given by the formula

$$(y_0 \oplus y_1)(x_0 \oplus x_1) = \langle x_0, y_0 \rangle_{X_0} + \langle x_1, y_1 \rangle_{X_1}.$$

Let Δ be the subspace of $X_0 \oplus_\infty X_1$ consisting of all elements of the form $x \oplus x$ for $x \in X_0 \cap X_1$ and equipped with the norm of $X_0 \oplus_\infty X_1$.

Let y be an arbitrary element of $(X_0 \cap X_1)^\#$. Then, for each $x \in X_0 \cap X_1$, we have $|\langle x, y \rangle| \le \|y\|_{(X_0 \cap X_1)^\#} \|x\|_{X_0 \cap X_1} = \|y\|_{(X_0 \cap X_1)^\#} \max\left\{\|x\|_{X_0}, \|x\|_{X_1}\right\}$. Thus the functional λ defined by $\lambda(x \oplus x) = \langle x, y \rangle$ for all $x \in X_0 \cap X_1$ defines a bounded linear functional on Δ with norm not exceeding $\|y\|_{(X_0 \cap X_1)^\#}$. By the Hahn–Banach theorem, there exists an element of $(X_0 \oplus_\infty X_1)^*$ which is an extension of λ and has the same norm. I.e., there exist $y_0 \in X_0^\#$ and $y_1 \in X_1^\#$ such that

$$\langle x, y \rangle = \langle x, y_0 \rangle_{X_0} + \langle x, y_1 \rangle_{X_1} \tag{7}$$

for all $x \in X_0 \cap X_1$ and $\|y_0\|_{X_0^\#} + \|y_1\|_{X_1^\#} = \|\lambda\|_{\Delta^*} \le \|y\|_{(X_0 \cap X_1)^\#}$. Since in (7) we are only considering elements x in $X_0 \cap X_1$, we can rewrite (7) as $\langle x, y \rangle = \langle x, y_0 \rangle + \langle x, y_1 \rangle = \langle x, y_0 + y_1 \rangle$. This shows that the element y can be written in the form $y = y_0 + y_1$, where $y_j \in X_j^\#$ for $j = 0, 1$ and so $y \in X_0^\# + X_1^\#$ with

$$\|y\|_{X_0^\# + X_1^\#} \le \|y_0\|_{X_0^\#} + \|y_1\|_{X_1^\#} \le \|y\|_{(X_0 \cap X_1)^\#}.$$

This establishes (6) and completes the proof of Fact 7. □

Fact 8. *Let X be a regular intermediate space with respect to the regular Banach couple (X_0, X_1). Then $X^\#$ is an intermediate space with respect to the Banach couple $(X_0^\#, X_1^\#)$.*

Proof. In view of Fact 3, the continuous inclusions $X_0 \cap X_1 \subset X \subset X_0 + X_1$ imply the continuous inclusions $(X_0 + X_1)^\# \subset X^\# \subset (X_0 \cap X_1)^\#$. We complete the proof by applying Fact 4 and Fact 7. □

Remark 9. *Suppose that U and V are both regular intermediate spaces with respect to the regular Banach couple (X_0, X_1). Then obviously both $U + V$ and $U \cap V$ are also intermediate spaces with respect to (X_0, X_1) and, clearly, $U + V$ is regular. In all examples that we know, the space $U \cap V$ is also regular. But it would be*

surprising if there are no examples where it is not regular, and we invite the reader to produce such an example. In view of our efforts on the first two pages of [5] *we are particularly curious to know if such an example can be found, or shown never to exist, in the special case where both U and V are complex interpolation spaces $U = [X_0, X_1]_{\theta_0}$ and $V = [X_0, X_1]_{\theta_1}$ (of course with $\theta_0 \neq \theta_1$).*

2. Calderón's theorem about duals of complex interpolation spaces

2.1. **Preface.** In this section we will discuss complex interpolation spaces $[X_0, X_1]_\theta$ as defined and studied by Alberto Calderón in [3]. For this purpose we will also need to be familiar with a variant of these spaces, denoted by $[X_0, X_1]^\theta$ and obtained by a construction which is sometimes referred to as Calderón's "second" or "upper" method. We refer to [3] Sections 2, 3, 5, 6 and 9.2 on pp. 114–116 for the definitions of $[X_0, X_1]_\theta$ and $[X_0, X_1]^\theta$ and for the definitions of the spaces $\mathcal{F}(X_0, X_1)$ and $\mathcal{G}(X_0, X_1)$ and $\overline{\mathcal{F}}(X_0, X_1)$ of Banach space valued analytic functions which are needed to define and study $[X_0, X_1]_\theta$ and $[X_0, X_1]^\theta$.

It will be convenient to use the notation $\mathbb{S}$ for the "unit strip" in the complex plane, i.e.,

$$\mathbb{S} = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$$

and to denote the interior of $\mathbb{S}$ by $\mathbb{S}^\circ$. The boundary $\partial\mathbb{S}$ of $\mathbb{S}$ is of course the union of the two vertical lines $\{it : t \in \mathbb{R}\}$ and $\{1 + it : t \in \mathbb{R}\}$.

We recall ([3] Sections 9.2–9.3 p. 116 and Sections 29.2–29.3 pp. 132–134) that $X_0 \cap X_1$ is dense in $[X_0, X_1]_\theta$ for each $\theta \in (0, 1)$. Thus the dual space of $[X_0, X_1]_\theta$ coincides with the space $([X_0, X_1]_\theta)^\#$ defined as in Section 1.

In the context and the notation that we have set up in Section 1, in particular bearing Fact 6 in mind, Calderón's remarkable duality theorem ([3] Section 12.1 p. 121) for his spaces $[X_0, X_1]_\theta$ can be expressed as follows:

Theorem 10. *Let (X_0, X_1) be a regular Banach couple. Then, for each $\theta \in (0, 1)$, the dual space $([X_0, X_1]_\theta)^\#$ coincides isometrically with $[X_0^\#, X_1^\#]^\theta$.*

In other words, the element $y \in (X_0 \cap X_1)^\#$ satisfies

$$\|y\|_{([X_0,X_1]_\theta)^\#} := \sup\left\{|\langle x, y\rangle| : x \in X_0 \cap X_1, \|x\|_{[X_0,X_1]_\theta} \leq 1\right\} < \infty$$

if and only if y is also an element of $[X_0^\#, X_1^\#]^\theta$. Furthermore, in that case,

$$\|y\|_{([X_0,X_1]_\theta)^\#} = \|y\|_{[X_0^\#,X_1^\#]^\theta}\,.$$

In keeping with the "pedantic" spirit of these notes, we are now going to give a proof of this theorem using the notions and notation of Section 1. Of course we are inspired by Calderón's beautiful and ingenious proof in [3] Section 32.1 pp. 148–156 and by the similar proof[1] presented in [1] pp. 98–101. But perhaps in some ways, despite its considerable length, the following proof might be considered simpler.

Here are some features of the proof to be presented here:

• In [3] and in [1] the inclusion $[X_0^\#, X_1^\#]^\theta \overset{1}{\subset} ([X_0, X_1]_\theta)^\#$ is proved using a somewhat elaborate multilinear interpolation theorem (Section 11.1 p. 120 and Section 31.1 on pp 142–148 of [3] and Theorem 4.4.2 on p. 97 of [1].) Here we

[1]The notation used in [1] is slightly different from that in [3].

can manage without this auxiliary theorem. However, instead, we have to use an apparently simpler (and very attractive!) result of Stafney.

• Calderón uses a connection between analytic functions on $\mathbb{S}^\circ$ and analytic functions on the open unit disk. In our proof the transition to analytic functions on the open unit disk will be made at an earlier stage. Some readers might like the fact that this approach saves us from having to deal with the Poisson kernels for $\mathbb{S}^\circ$ which are given by somewhat complicated formulæ (which can be seen, for example, on p. 117 of [3]). Indeed we have set ourselves the goal of using only the simplest possible forms of various results about scalar valued analytic functions. Thus, for example, we essentially avoid having to talk about boundary behaviour of bounded analytic functions, nontangential limits, harmonic conjugates of arbitrary smooth functions, etc.

• We will use an alternative and in some ways more convenient definition of the space $[Y_0, Y_1]^\theta$ which is available in the particular case, the only case that we need here, where $(Y_0, Y_1) = (X_0^\#, X_1^\#)$, i.e., where (Y_0, Y_1) is a couple of dual spaces of some regular couple (X_0, X_1). This alternative definition is analogous to the definition of the space denoted by $B^+\{z_0\}^*$ in [4] pp. 211–212 and indeed some parts of our proof here are suggested by things in [4]. (In the setting of [4] where one deals with n–tuples or infinite families of Banach spaces instead of couples, there is apparently no alternative to using this kind of alternative definition of an "upper" method.)

2.2. **Preliminaries about scalar valued analytic functions.** A number of rather standard results about scalar valued harmonic and analytic functions will be used in our proof. I prefer that we discuss them now, in advance, and in considerable detail, before we start that proof. This is one place where I might particularly recall my apology at the beginning of these notes.

As usual $\mathbb{T}$ denotes the unit circle, $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$. We also let $\mathbb{D}$ denote the closed unit disk $\mathbb{D} = \{w \in \mathbb{C} : |w| \le 1\}$ and of course $\mathbb{D}^\circ$ denotes its interior.

There are of course conformal maps of $\mathbb{D}^\circ$ onto $\mathbb{S}^\circ$. We will find it convenient to use a particular one of these, which we will denote by $\xi : \mathbb{D}^\circ \to \mathbb{S}^\circ$, and whose choice depends on the value of the parameter $\theta \in (0, 1)$ appearing in the statement of Theorem 10.

For the (rather standard) construction of ξ we first consider the Möbius transformation

$$\mu(w) = \frac{we^{-i\pi\theta} - e^{i\pi\theta}}{w - 1}, \tag{8}$$

which maps 1 to ∞, $e^{i2\pi\theta}$ to 0 and each other point on the unit circle $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$ to a non zero real number. More explicitly, each $e^{it} \in \mathbb{T}\backslash\{1, e^{i2\pi\theta}\}$ is mapped to

$$\mu(e^{it}) = \frac{e^{i(t-\pi\theta)} - e^{i\pi\theta}}{e^{it} - 1} = \frac{e^{i(t/2-\pi\theta)} - e^{i(\pi\theta-t/2)}}{e^{it/2} - e^{-it/2}} = \frac{\sin(t/2 - \pi\theta)}{\sin t/2}. \tag{9}$$

We introduce the two subintervals $I_0 := (2\pi\theta, 2\pi)$ and $I_1 := (0, 2\pi\theta)$ of $(0, 2\pi)$, and the two arcs Γ_0 and Γ_1 defined by $\Gamma_j = \left\{e^{it} : t \in I_j\right\}$. We see from (9) that μ maps Γ_0 onto the positive real axis, and Γ_1 onto the negative real axis.

We also have $\mu(0) = e^{i\pi\theta}$. Thus, since $\theta \in (0, 1)$, the transformation μ conformally maps the open unit disk $\mathbb{D}^\circ = \{w \in \mathbb{C} : |w| < 1\}$ onto the open upper half

plane and is analytic and conformal at every point $w \neq 1$ of the entire complex plane. As in [3], this leads us to introduce the map

$$\xi(w) = \frac{1}{i\pi} \log\left(\frac{we^{-i\pi\theta} - e^{i\pi\theta}}{w-1}\right), \tag{10}$$

where here we choose a branch of the complex logarithm which is analytic and conformal at every non zero point of the closed upper half plane, and whose imaginary part lies in the range $[0, \pi]$ at every such point. It follows that ξ maps $\mathbb{D}^\circ$ conformally onto $\mathbb{S}^\circ$ and, furthermore, is also analytic and conformal at every point of $\mathbb{D}\backslash\{1, e^{i2\pi\theta}\}$. We note that $\xi(0) = \theta$ and that the images of the arcs Γ_0 and Γ_1 satisfy

$$\xi(\Gamma_j) = \{j + is : s \in \mathbb{R}\} \text{ for } j = 0, 1. \tag{11}$$

Let us also write down the inverse map of ξ: If $\xi(w) = z$ then $\frac{we^{-i\pi\theta}-e^{i\pi\theta}}{w-1} = e^{i\pi z}$ and, consequently, $w = \xi^{-1}(z) := \frac{e^{i\pi\theta}-e^{i\pi z}}{e^{-i\pi\theta}-e^{i\pi z}}$. So we see that the function ξ^{-1} is analytic at every point $z \in \mathbb{C}$ which is not an element of the sequence $\{2k - \theta\}_{k\in\mathbb{Z}}$. More relevantly for us, it maps $\mathbb{S}^\circ$ conformally onto $\mathbb{D}^\circ$, and is also conformal at every point of $\partial\mathbb{S}$.

We recall the following standard result of Phragmen–Lindelöf type, often referred to as the "three lines theorem" and associated with the names of Hadamard and Doetsch. (More precisely, the analogous "three circles theorem" is due to Jacques Hadamard and its variant for three lines is due to Gustav Doetsch.) We state it here in a version which is adequate for our particular needs. (In fact (12) can be replaced by a very much weaker condition.)

Lemma 11. *Let M_0, M_1 and c be positive constants.*

(i) Suppose that $f : \mathbb{S} \to \mathbb{C}$ is a continuous function which is analytic on $\mathbb{S}^\circ$ and satisfies

$$|f(z)| \leq c(1 + |z|) \text{ for all } z \in \mathbb{S}^\circ \tag{12}$$

and

$$\sup_{t\in\mathbb{R}} |f(j + it)| \leq M_j \text{ for } j = 0, 1. \tag{13}$$

Then

$$|f(z)| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z} \text{ for all } z \in \mathbb{S}^\circ. \tag{14}$$

(ii) The same conclusion (14) holds if f is not defined on $\partial\mathbb{S}$ and is merely an analytic function on $\mathbb{S}^\circ$ which, instead of condition (13), satisfies

$$\lim_{r\searrow 0} \sup\{|f(z)| : z \in \mathbb{S},\ 0 < |\operatorname{Re} z - j| < r\} \leq M_j \text{ for } j = 0, 1. \tag{15}$$

Proof. It is almost quicker to give a proof than a reference, and, anyway, parts of the proof here will also be relevant later.

We first deal with part (i): Let δ be a positive constant and consider the entire function $\phi(z) = \exp(\delta z^2 - (1-z)\log M_0 - z \log M_1)$. Fix $z_* \in \mathbb{S}^\circ$. By the maximum modulus principle, $|\phi(z_*)f(z_*)| \leq \sup_{z\in\partial E_R} |\phi(z)f(z)|$ where E_R is the rectangle $\{z \in \mathbb{S} : |\operatorname{Im} z| \leq R\}$ and $R > |z_*|$. In view of

$$\left|e^{\delta z^2}\right| = e^{\delta[(\operatorname{Re} z)^2 - (\operatorname{Im} z)^2]} \tag{16}$$

and (13) we see that $|\phi(j+it)f(j+it)| \le e^{\delta j}$ for $j=0,1$ and all $t \in \mathbb{R}$ and, in view of (12) and (16), we can also ensure, by choosing R sufficiently large, that $|\phi(z)f(z)| \le 1$ at every point z on each of the two horizontal line segments of ∂E_R. We deduce that $|\phi(z_*)f(z_*)| \le e^{\delta}$. So $|f(z_*)| \le e^{\delta}/|\phi(z_*)| = |e^{\delta(1-z_*^2)}| M_0^{1-\operatorname{Re} z_*} M_1^{\operatorname{Re} z_*}$. Since we can choose δ arbitrarily small, this implies (14).

Part (ii) is a straightforward corollary of part (i). In this case, given an analytic function $f : \mathbb{S}^\circ \to \mathbb{C}$ which satisfies (12) and (15), we simply apply part (i) to the function $f_r(z) := f\left(r/2 + (1-r)z\right)$ where $r \in (0, 1/2)$ is a constant. It is clear that f_r satisfies all the hypotheses of part (i), with c replaced by some other constant c_r depending on r, and M_j replaced by $\sup_{t\in\mathbb{R}} |f(|j - r/2| + it)|$. Thus part (i) gives us, for each $z \in \mathbb{S}^\circ$, that

$$|f_r(z)| \le \left(\sup\left\{|f(\zeta)| : 0 < \operatorname{Re}\zeta < r\right\}\right)^{1-\operatorname{Re} z} \cdot \left(\sup\left\{|f(\zeta)| : 0 < 1 - \operatorname{Re}\zeta < r\right\}\right)^{\operatorname{Re} z}$$

and we obtain (14) for f by passing to the limit as r tends to 0. □

Remark 12. *A standard example shows that the conclusions of Lemma 11 may fail to hold if the condition (12) is not imposed. However, as already mentioned, (12) can be replaced by much weaker conditions.*

For any open connected set $U \subset \mathbb{C}$ we define $H^\infty(U)$ to be the set of all bounded analytic functions $f : U \to \mathbb{C}$ with norm $\|f\|_{H^\infty(U)} = \sup_{z\in U} |f(z)|$. Here we are only interested in the particular spaces $H^\infty(\mathbb{D}^\circ)$ and $H^\infty(\mathbb{S}^\circ)$.

Of course $f \in H^\infty(\mathbb{S}^\circ)$ if and only if $g = f \circ \xi$ is a function in $H^\infty(\mathbb{D}^\circ)$, and the map $f \mapsto f \circ \xi$ and its inverse $g \mapsto g \circ \xi^{-1}$ define isometries between $H^\infty(\mathbb{S}^\circ)$ and $H^\infty(\mathbb{D}^\circ)$.

The following theorem is a combination of well known facts about Fourier series (see e.g., [12] Chapter 1) and the Poisson integral for the disk (see e.g., [14] Chapter 11). It is only a "light" or "lite" version, sufficient for our needs here, and we have provided an almost self contained proof. Much more could be said, e.g., about the existence of radial or even nontangential limits of the function u and their almost everywhere coincidence with the values of h. Among other things, part (vi) of the theorem generalizes Lemma 11.

Theorem 13. *Suppose that the measurable function $h : \mathbb{T} \to \mathbb{C}$ satisfies*

$$\int_0^{2\pi} |h(e^{it})| dt < \infty \; . \tag{17}$$

Set $\widehat{h}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int} h(e^{it}) dt$ for each $n \in \mathbb{Z}$. For each $z = re^{it}$ with $r \ge 0$ and $t \in \mathbb{R}$ define

$$u(z) = \sum_{n=-\infty}^{\infty} \widehat{h}(n) r^{|n|} e^{int} \tag{18}$$

Then:

(i) The series $\sum_{n=-\infty}^{\infty} \widehat{h}(n) r^{|n|} e^{int}$ converges absolutely for every $r \in [0,1)$ and $t \in \mathbb{R}$ and thus (18) defines a continuous function $u : \mathbb{D}^\circ \to \mathbb{C}$.

(ii) The function $u : \mathbb{D}^\circ \to \mathbb{C}$ is harmonic in $\mathbb{D}^\circ$.

(iii) If $\widehat{h}(n) = 0$ for every $n < 0$ then u is also analytic in $\mathbb{D}^\circ$.

(iv) If h is essentially bounded then

$$|u(z)| \leq \operatorname*{ess\,sup}_{t\in[0,2\pi]} \left|h(e^{it})\right| \text{ for all } z \in \mathbb{D}^{\circ}.$$

(v) Suppose that $f : \mathbb{D} \to \mathbb{C}$ is a continuous function which is analytic in $\mathbb{D}^{\circ}$ and $\widehat{h}(n) = 0$ for every $n < 0$, and $g(e^{it}) = h(e^{it})f(e^{it})$ for a.e. $t \in [0, 2\pi]$ with Fourier coefficients $\widehat{g}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}g(e^{it})dt$. Then

$$\sum_{n=-\infty}^{\infty} \widehat{g}(n)r^{|n|}e^{int} = u(re^{it})f(re^{it}) \text{ for every } r \in [0, 1) \text{ and } t \in \mathbb{R},$$

and, in particular,

$$\frac{1}{2\pi}\int_0^{2\pi} g(e^{it})dt = u(0)f(0).$$

(vi) Let Γ_0 and Γ_1 and ξ be the two arcs and the conformal map defined above. Let M_0 and M_1 be positive constants. Suppose that $\left|h(e^{it})\right| \leq M_0$ for almost all $e^{it} \in \Gamma_0$ and $\left|h(e^{it})\right| \leq M_1$ for almost all $e^{it} \in \Gamma_1$ and that $\widehat{h}(n) = 0$ for every $n < 0$. Then

$$\left|u(\xi^{-1}(z))\right| \leq M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z} \text{ for all } z \in \mathbb{S}^{\circ}. \tag{19}$$

Remark 14. *In fact in our subsequent applications we will not really need the fact (conclusion (ii) above) that u is harmonic.*

Proof. Properties (i), (ii) and (iii) are rather obvious. Property (i) is a trivial consequence of the boundedness of the sequence $\left\{\widehat{h}(n)\right\}_{n\in\mathbb{Z}}$. It implies in turn that the convergence of $u_N(re^{it}) := \sum_{n=-N}^{N} \widehat{h}(n)r^{|n|}e^{int}$ to $u(re^{it})$ as N tends to ∞ is uniform on every compact subset of $\mathbb{D}^{\circ}$. Since each u_N is harmonic, it follows by Harnack's theorem (see e.g., [14] p. 236, Theorem 11.11) that u is harmonic on $\mathbb{D}^{\circ}$. This establishes (ii). Property (iii) follows in almost the same way since, under the stated condition, the functions u_N will also be analytic.

To obtain (iv) we first note that, for each fixed $r \in [0, 1)$ and $t \in \mathbb{R}$, the uniform convergence of $s \mapsto \sum_{n=-\infty}^{\infty} r^{|n|}e^{in(t-s)}$ on $\mathbb{R}$ permits us to integrate term by term in the following calculation to obtain that

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \widehat{h}(n)r^{|n|}e^{int} &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} h(e^{is})r^{|n|}e^{in(t-s)}ds \\
&= \frac{1}{2\pi}\int_0^{2\pi} h(e^{is})\left[\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(t-s)}\right]ds. \qquad (20)
\end{aligned}$$

Although I have been trying obstinately to avoid dealing explicitly with the Poisson kernel for the disk, it seems that I have no alternative here but to carry out the (admittedly very simple) calculation (see e.g., [14] p. 111) which shows that

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(t-s)} = \frac{1-r^2}{1-2r\cos(t-s)+r^2}. \tag{21}$$

The only reason that we need (21) is in order to be able to affirm that

$$\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(t-s)} \geq 0 \text{ for all } r \in [0,1) \text{ and } t \in \mathbb{R}\,.$$

By considering (20) in the special case where $h(e^{it}) = 1$ for all t, we see that

$$\frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(t-s)}\right]ds = 1.$$

So

$$\begin{aligned}
\left|u(re^{it})\right| &= \left|\frac{1}{2\pi}\int_0^{2\pi} h(e^{is})\left[\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(t-s)}\right]ds\right| \\
&\leq \frac{1}{2\pi}\int_0^{2\pi} |h(e^{is})|\cdot\left|\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(t-s)}\right|ds \\
&\leq \operatorname*{ess\,sup}_{t\in[0,2\pi]}\left|h(e^{it})\right|\cdot\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(t-s)}\right|ds = \operatorname*{ess\,sup}_{t\in[0,2\pi]}\left|h(e^{it})\right|
\end{aligned}$$

as required.

To prove (v) we start by rewriting the product $u(re^{it})f(re^{it})$ as a double sum

$$\begin{aligned}
u(re^{it})f(re^{it}) &= \sum_{n=0}^{\infty}\widehat{h}(n)r^n e^{int}\cdot\sum_{m=0}^{\infty}\widehat{f}(m)r^m e^{imt} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\widehat{h}(n)\widehat{f}(m)r^{m+n}e^{i(m+n)t}\right). \qquad (22)
\end{aligned}$$

Since $\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}|\widehat{h}(n)\widehat{f}(m)r^{(m+n)}e^{i(m+n)t}|\right) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}|\widehat{h}(n)||\widehat{f}(m)|r^{m+n}\right)$ is finite, we can change the order of summation in (22) so that in fact

$$\begin{aligned}
u(re^{it})f(re^{it}) &= \sum_{k=0}^{\infty}\left(\sum_{m+n=k,m\geq 0,n\geq 0}\widehat{h}(n)\widehat{f}(m)r^{m+n}e^{i(m+n)t}\right) \\
&= \sum_{k=0}^{\infty}\left(\sum_{m+n=k,m\geq 0,n\geq 0}\widehat{h}(n)\widehat{f}(m)\right)r^k e^{ikt}.
\end{aligned}$$

To complete the proof we have to show that, for each $k \in \mathbb{Z}$,

$$\widehat{g}(k) = \begin{cases} \displaystyle\sum_{m+n=k,m\geq 0,n\geq 0}\widehat{h}(n)\widehat{f}(m) & , \quad k \geq 0 \\ 0 & , \quad k < 0 \end{cases} \qquad (23)$$

We first observe that $\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}f(e^{it})dt = \frac{1}{2\pi i}\int_0^{2\pi} e^{-i(n+1)t}f(e^{it})ie^{it}dt$ is exactly the complex contour integral $\frac{1}{2\pi i}\oint_{\mathbb{T}} z^{-n-1}f(z)dz$ and so, by Cauchy's theorem, it vanishes for every negative integer n. For each $k \in \mathbb{Z}$, we will calculate $\widehat{g}(k)$ with the help of the function $v : \mathbb{T} \to \mathbb{C}$ defined by $v(e^{it}) = e^{ikt}\overline{f(e^{it})}$. The Fourier coefficients of this function are $\widehat{v}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n-k)}\overline{f(e^{it})}dt = \overline{\widehat{f}(k-n)}$.

Both $|h|$ and $|v|$ are essentially bounded and therefore square integrable on $\mathbb{T}$. So we can apply the generalized Parseval identity to obtain that

$$\begin{aligned}\widehat{g}(k) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt}h(e^{it})f(e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})\overline{v(e^{it})}dt = \sum_{n\in\mathbb{Z}}\widehat{h}(n)\overline{\widehat{v}(n)} \\ &= \sum_{n\in\mathbb{Z}}\widehat{h}(n)\widehat{f}(k-n).\end{aligned}$$

Since the summands in this series vanish for all $n<0$ and all $n>k$, this establishes (23) for each $k\in\mathbb{Z}$ and so completes the proof of part (v).

For part (vi) we will once more use the function $\phi:\mathbb{C}\to\mathbb{C}$ which we already used in the proof of Lemma 11. I.e., we have $\phi(z)=\exp(\delta z^2-(1-z)\log M_0 - z\log M_1)$ where δ is a positive constant. Since $\phi(z)$ is analytic at every point $z\in\mathbb{C}$, it follows that the function $f(w):=\phi(\xi(w))$ is analytic and non zero at every point $w\in\mathbb{D}\backslash\{1,e^{i2\pi\theta}\}$.

Let $\{w_n\}_{n\in\mathbb{N}}$ be any sequence of points in $\mathbb{D}\backslash\{1,e^{i2\pi\theta}\}$ which tends to either 1 or $e^{i2\pi\theta}$. Then the sequence $\{\mu(w_n)\}_{n\in\mathbb{N}}$, where μ is the Möbius transformation (8), lies in the closed upper half plane and must tend to either ∞ or 0. Consequently, the sequence $\{\xi(w_n)\}_{n\in\mathbb{N}}$ lies in $\mathbb{S}$ and $\operatorname{Im}\xi(w_n)$ must tend to either $+\infty$ or $-\infty$. So, using (16), we see that f extends to a continuous function on $\mathbb{D}$ with $f(1)=f(e^{i2\pi\theta})=0$. I.e., it satisfies the hypotheses of (v). Furthermore (recalling (11)) we see that, for $j=0,1$,

$$\sup_{e^{it}\in\Gamma_j}|f(e^{it})| = \sup_{t\in\mathbb{R}}|\phi(j+it)| = e^{\delta j}/M_j\ . \tag{24}$$

It follows from (24) and our hypotheses on h, that the function $g(e^{it}):=h(e^{it})f(e^{it})$ satisfies ess $\sup_{t\in[0,2\pi]}|g(e^{it})|\le e^{\delta}$. Thus, applying part (iv), but to the function g instead of h, and then applying part (v), we see that

$$e^{\delta}\ge\left|\sum_{n=-\infty}^{\infty}\widehat{g}(n)r^{|n|}e^{int}\right| = |u(re^{it})f(re^{it})| \text{ for every } r\in[0,1) \text{ and } t\in\mathbb{R}.$$

Given any $z\in\mathbb{S}^\circ$, choose $r\in[0,1)$ and $t\in\mathbb{R}$ such that $z=\xi(re^{it})$. Then

$$\left|u(\xi^{-1}(z))\right| = \left|u(re^{it})\right| \le \frac{e^{\delta}}{|f(re^{it})|} = \frac{e^{\delta}}{|\phi(z)|} = \frac{e^{\delta}}{|e^{\delta z^2}|M_0^{-1+\operatorname{Re}z}M_1^{-\operatorname{Re}z}}.$$

Since we can choose the positive number δ to be as small as we please, these estimates immediately imply (19) and complete the proof of (vi) and thus of Theorem 13. □

We will also need another result about functions in $H^\infty(\mathbb{S}^\circ)$:

Lemma 15. *Let h be a function in $H^\infty(\mathbb{S}^\circ)$. Then there exists a function $\phi:\mathbb{S}\to\mathbb{C}$ with the following properties:*

(i) $|\phi(z_1)-\phi(z_2)|\le|z_1-z_2|\cdot\sup_{\zeta\in\mathbb{S}^\circ}|h(\zeta)|$ *for all* $z_1,z_2\in\mathbb{S}$.

(ii) ϕ *is analytic in* $\mathbb{S}^\circ$ *and satisfies* $\phi'(z)=h(z)$ *for each* $z\in\mathbb{S}^\circ$,

(iii) The estimate

$$|\phi(j+it_1)-\phi(j+it_2)|\le|t_1-t_2|\cdot\lim_{r\searrow 0}\left(\sup\{|h(z)|: z\in\mathbb{S},\ 0<|\operatorname{Re}z-j|<r\}\right)$$

holds for $j=0,1$ *and all* $t_1,t_2\in\mathbb{R}$.

(iv) $|\phi(z)|\le\left|z-\frac{1}{2}\right|\cdot\sup_{\zeta\in\mathbb{S}^\circ}|h(\zeta)|$ *for all* $z\in\mathbb{S}$.

Proof. We define $\phi(z)$ for all $z \in \mathbb{S}$ by the two formulæ

$$\phi(z) = \int_{1/2}^{z} h(\zeta)d\zeta \text{ for each } z \in \mathbb{S}^\circ \tag{25}$$

and

$$\phi(j+it) = \lim_{n\to\infty} \int_{1/2}^{j+it+(1/2-j)n} h(\zeta)d\zeta \text{ for each } j = 0,1 \text{ and } t \in \mathbb{R}. \tag{26}$$

In (25) the contour integration is performed along a contour, any contour, in $\mathbb{S}^\circ$ from $1/2$ to z. This ensures that property (ii) holds. We obtain (i), at least in the case where z_1 and z_2 are both in $\mathbb{S}^\circ$, by choosing a contour from $1/2$ to z_1 which includes a straight line segment from z_2 to z_1. This shows that ϕ is uniformly continuous on $\mathbb{S}^\circ$ and therefore we can extend our definition of ϕ to all of $\mathbb{S}$. This will of course be the unique continuous extension of ϕ to $\mathbb{S}$ whose values on $\partial\mathbb{S}$ must necessarily equal those given by the formula (26). It is now clear that ϕ satisfies (i) on all of $\mathbb{S}$, and property (iv) follows obviously from property (i) together with the fact that $\phi(1/2) = 0$.

It remains to check that property (iii) also holds: For $j = 0,1$ and for each real t_1 and t_2 we have

$$\begin{aligned}\phi(j+it_2) - \phi(j+it_1) &= \lim_{n\to\infty} \phi\left(j+it_2+\frac{1/2-j}{n}\right) - \phi\left(j+it_1+\frac{1/2-j}{n})\right) \\ &= i\lim_{n\to\infty}\int_{t_1}^{t_2} h\left(j+\frac{1/2-j}{n}+it\right)dt\ . \end{aligned}\tag{27}$$

Given $\epsilon > 0$, we can choose $r_\epsilon > 0$ such that

$$\begin{aligned}&\sup\{|h(z)| : z \in \mathbb{S},\ 0 < |\operatorname{Re} z - j| < r_\epsilon\} \\ \leq\ &\epsilon + \lim_{r\searrow 0}\left(\sup\{|h(z)| : z \in \mathbb{S},\ 0 < |\operatorname{Re} z - j| < r\}\right).\end{aligned}$$

So, for all $n > 1/r_\epsilon$, we have

$$|h(j+(1/2-j)/n+it)| \leq \epsilon + \lim_{r\searrow 0}\left(\sup\{|h(z)| : z \in \mathbb{S},\ 0 < |\operatorname{Re} z - j| < r\}\right).$$

This, combined with (27), shows that

$$|\phi(j+it_2) - \phi(j+it_1)| \leq |t_1 - t_2|\left(\epsilon + \lim_{r\searrow 0}\left(\sup\{|h(z)| : z \in \mathbb{S},\ 0 < |\operatorname{Re} z - j| < r\}\right)\right).$$

Since ϵ can be chosen arbitrarily small, we obtain property (iii). □

2.3. An alternative definition of the space $[X_0, X_1]_\theta$ and another preliminary result.

Definition 16. *We define $\mathcal{F}_{\mathbb{D}}(X_0, X_1)$ to be the space of all continuous functions $f : \mathbb{D} \to X_0 + X_1$ which are analytic in $\mathbb{D}^\circ$ and satisfy $f(1) = f(e^{i2\pi\theta}) = 0$ and such that, for $j = 0,1$, the restriction of f to the closed arc $\overline{\Gamma_j}$ is a continuous X_j valued function. We norm this space by*

$$\|f\|_{\mathcal{F}_{\mathbb{D}}(X_0,X_1)} = \max\left\{\max_{z\in\Gamma_0}\|f(z)\|_{X_0}, \max_{z\in\Gamma_1}\|f(z)\|_{X_1}\right\}. \tag{28}$$

Obviously, any given function $g : \mathbb{S} \to X_0 + X_1$ is an element of $\mathcal{F}(X_0, X_1)$ if and only if the function $g \circ \xi : \mathbb{D} \backslash \{1, e^{i2\pi\theta}\} \to X_0 + X_1$, can be extended continuously to all of $\mathbb{D}$ and this extension is an element of $\mathcal{F}_{\mathbb{D}}(X_0, X_1)$. (Here ξ is of course the function defined by (10).) In that case we also have $\|g\|_{\mathcal{F}(X_0,X_1)} = \|g \circ \xi\|_{\mathcal{F}_{\mathbb{D}}(X_0,X_1)}$. Furthermore, since $\xi(0) = \theta$, we deduce immediately that

$$[X_0, X_1]_\theta = \{f(0) : f \in \mathcal{F}_{\mathbb{D}}(X_0, X_1)\} \tag{29}$$

and

$$\|x\|_{[X_0,X_1]_\theta} = \inf\left\{\|f\|_{\mathcal{F}_{\mathbb{D}}(X_0,X_1)} : f \in \mathcal{F}_{\mathbb{D}}(X_0, X_1), f(0) = x\right\}. \tag{30}$$

Calderón's space $\mathcal{G}(X_0, X_1)$, defined on p. 116 of [3], is a very useful dense subspace of $\mathcal{F}(X_0, X_1)$. We will need a more or less analogous dense subspace of $\mathcal{F}_{\mathbb{D}}(X_0, X_1)$.

Definition 17. *Let $\mathcal{G}_{\mathbb{D}}(X_0, X_1)$ be the subspace of $\mathcal{F}_{\mathbb{D}}(X_0, X_1)$ consisting of all elements $f : \mathbb{D} \to X_0 \cap X_1$ which are finite sums of the form*

$$f(w) = \sum_{k=1}^{N} \psi_k(w) a_k \tag{31}$$

where each $a_k \in X_0 \cap X_1$ and each $\psi_k : \mathbb{D} \to \mathbb{C}$ is a continuous function on $\mathbb{D}$ which is analytic in $\mathbb{D}^\circ$ and satisfies $\psi_k(1) = \psi_k(e^{i2\pi\theta}) = 0$.

Remark 18. *Our definition here gives a somewhat larger class than would be obtained by simply taking the image of $\mathcal{G}(X_0, X_1)$ under composition with the function ξ. In fact the class of functions*

$$f(w) = \begin{cases} g(\xi(w)) & , \quad w \in \mathbb{D}\backslash\{1, e^{i2\pi\theta}\} \\ 0 & , \quad w = 1, e^{i2\pi\theta} \end{cases},$$

where $g : \mathbb{S} \to X_0 \cap X_1$ is an element of $\mathcal{G}(X_0, X_1)$, is precisely the subspace of $\mathcal{G}_{\mathbb{D}}(X_0, X_1)$ where the functions ψ_k in (31) are each required to be of the special form $\psi_k(w) = e^{\delta_k \xi(w)^2 + i\lambda_k \xi(w)}$ for some constants $\delta_k > 0$ and $\lambda_k \in \mathbb{R}$ for all $w \in \mathbb{D}\backslash\{1, e^{i2\pi\theta}\}$. This class is dense in $\mathcal{F}_{\mathbb{D}}(X_0, X_1)$ because of the density (see [3] *Section 9.2 p. 116 and Section 29.2 pp. 132–133) of $\mathcal{G}(X_0, X_1)$ in $\mathcal{F}(X_0, X_1)$. So of course $\mathcal{G}_{\mathbb{D}}(X_0, X_1)$ must also be dense in $\mathcal{F}_{\mathbb{D}}(X_0, X_1)$.*

Remark 19. *Our definitions here of $\mathcal{F}_{\mathbb{D}}(X_0, X_1)$ and $\mathcal{G}_{\mathbb{D}}(X_0, X_1)$ depend of course on our choice of the parameter $\theta \in (0, 1)$. In some contexts, for example where, unlike here, it is necessary to simultaneously consider different values of θ, it might be more natural to first define ξ and $\mathcal{F}_{\mathbb{D}}(X_0, X_1)$ for the particular choice $\theta = 1/2$. In that case, instead of (29) and (30) we would have $[X_0, X_1]_\alpha = \left\{f(\xi^{-1}(\alpha)) : f \in \mathcal{F}_{\mathbb{D}}(X_0, X_1)\right\}$ and*

$$\|x\|_{[X_0,X_1]_\alpha} = \inf\left\{\|f\|_{\mathcal{F}_{\mathbb{D}}(X_0,X_1)} : f \in \mathcal{F}_{\mathbb{D}}(X_0, X_1), f(\xi^{-1}(\alpha)) = x\right\}$$

for each $\alpha \in (0, 1)$.

The following result is essentially equivalent to the third inequality stated in Section 9.4 on p. 117 of [3], and proved in Section 29.4 on pp. 134–135 of [3]. But

we offer the reader a self contained and perhaps in some ways simpler[2] proof using almost the same approach as in [3]. (Cf. also [4] Proposition 2.4 pp. 209–210).

Lemma 20. *The inequality*

$$\|f(0)\|_{[X_0,X_1]_\theta} \le \frac{1}{2\pi}\left(\int_{2\pi\theta}^{2\pi}\left\|f(e^{it})\right\|_{X_0}dt + \int_0^{2\pi\theta}\left\|f(e^{it})\right\|_{X_1}dt\right) \tag{32}$$

holds for every $f \in \mathcal{F}_{\mathbb{D}}(X_0,X_1)$.

Proof. It will be notationally convenient to introduce the continuous function $h:\mathbb{T}\to[0,\infty)$ defined by

$$h(e^{it}) = \begin{cases} \left\|f(e^{it})\right\|_{X_0} & , \quad 2\pi\theta < t < 2\pi \\ \left\|f(e^{it})\right\|_{X_1} & , \quad 0 < t < 2\pi\theta \\ 0 & , \quad t = 0,\ 2\pi\theta. \end{cases} \tag{33}$$

and so to rewrite (32) as

$$\|f(0)\|_{[X_0,X_1]_\theta} \le \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})dt. \tag{34}$$

By Jensen's inequality ([14] Theorem 3.3 p. 62) and the convexity of the exponential function, we have

$$\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left(h(e^{it})+\rho\right)dt\right) \le \frac{1}{2\pi}\left(\int_0^{2\pi}\left(h(e^{it})+\rho\right)dt\right) \tag{35}$$

for each positive number ρ. So our main step will be to prove that

$$\|f(0)\|_{[X_0,X_1]_\theta} \le \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left(h(e^{it})+\rho\right)dt\right). \tag{36}$$

for each $\rho > 0$. Then we can obtain (34) by simply substituting in (35), and then taking the limit as ρ tends to 0.

Choose an arbitrary positive number ϵ. Since $e^{it} \mapsto \log\left(h(e^{it})+\rho\right)$ is a continuous real valued function on $\mathbb{T}$, there exists (cf. e.g., [12] p. 15) a real valued trigonometric polynomial $p(e^{it}) = \sum_{n=-N}^{N} c_n e^{int}$ such that

$$\log\left(h(e^{it})+\rho\right) \le p(e^{it}) + \epsilon \le \log\left(h(e^{it})+\rho\right) + 2\epsilon \tag{37}$$

for all $e^{it} \in \mathbb{T}$. The fact that our polynomial is real valued implies that $\overline{c_n} = c_{-n}$ for each n. A trivial calculation using this last formula shows that the functions $u:\mathbb{C}\to\mathbb{C}$ and $v:\mathbb{C}\to\mathbb{C}$ defined by

$$u(re^{it}) = \sum_{n=-N}^{N} c_n r^{|n|} e^{it} \text{ and } v(re^{it}) = \frac{1}{i}\sum_{n=1}^{N} c_n r^{|n|} e^{int} - \frac{1}{i}\sum_{n=-N}^{-1} c_n r^{|n|} e^{int}$$

for all $r \ge 0$ and $t \in \mathbb{R}$ are both real valued. It is also clear that the function $w:\mathbb{C}\to\mathbb{C}$ defined by $w(z) = u(z) + iv(z)$ is a polynomial in z and thus analytic for all z.

---

[2]One thing which might be considered a simplification here is that, in contrast to [3] and [4], we only have to consider the obvious and explicit complex conjugates of harmonic polynomials instead of complex conjugates of more general smooth functions.

This last fact ensures that the function $g(z) := e^{-w(z)-\epsilon}f(z)$ is an element of $\mathcal{F}_{\mathbb{D}}(X_0, X_1)$. Furthermore, for each $e^{it} \in \mathbb{T}$ we have

$$\left|e^{-w(z)-\epsilon}\right| = e^{-u(e^{it})-\epsilon} = e^{-p(e^{it})-\epsilon} \leq e^{-\log(h(e^{it})+\rho)} = \frac{1}{h(e^{it})+\rho}.$$

Thus, for $t \in (2\pi\theta, 2\pi)$, i.e., for $z = e^{it} \in \Gamma_0$, we have, recalling (33), that

$$\|g(z)\|_{X_0} = \left|e^{-w(z)-\epsilon}\right| \left\|f(e^{it})\right\|_{X_0} \leq \frac{\left\|f(e^{it})\right\|_{X_0}}{\|f(e^{it})\|_{X_0}+\rho} \leq 1.$$

Exactly analogous considerations show that $\left\|g(e^{it})\right\|_{X_1} \leq 1$ for each $e^{it} \in \Gamma_1$ and so we have $\|g(0)\|_{[X_0,X_1]_\theta} \leq \|g\|_{\mathcal{F}_{\mathbb{D}}(X_0,X_1)} \leq 1$. This means that

$$\|f(0)\|_{[X_0,X_1]_\theta} = \left|e^{w(0)+\epsilon}\right| \|g(0)\|_{[X_0,X_1]_\theta} \leq \left|e^{w(0)+\epsilon}\right| = e^{u(0)+\epsilon}. \tag{38}$$

Since u is a harmonic function, (or, even more simply, since $u(0) = c_0$ and $\int_0^{2\pi} e^{int}dt = 0$ for every non zero $n \in \mathbb{Z}$) we have $u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})dt = \frac{1}{2\pi}\int_0^{2\pi} p(e^{it})dt$. This, combined with (38) and (37) shows that

$$\|f(0)\|_{[X_0,X_1]_\theta} \leq \exp\left(\frac{1}{2\pi}\int_0^{2\pi}(\log(h(e^{it})+\rho)+\epsilon)dt+\epsilon\right).$$

Since ϵ can be chosen arbitrarily small, this gives us (36) and so completes the proof of the lemma. □

We conclude this subsection with a lemma which in fact is not required for the proof of Theorem 10. We will probably need it, or rather its almost immediate corollary, in a paper which is currently in preparation. The lemma is a very special case, but one sufficient for our purposes in that paper, of a quite standard result, namely, Theorem 15.19 of [14] p. 308. We waited till now, rather than including it earlier in Subsection 2.2, since its proof can be almost immediately deduced from part of the proof of Lemma 20.

Lemma 21. *Suppose that the function $\psi : \mathbb{D} \to \mathbb{C}$ is continuous and is analytic in $\mathbb{D}^\circ$. Suppose further that $\psi(1) = \psi(e^{i2\pi\theta}) = 0$. Then*

$$|\psi(0)| \leq \exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left|\psi(e^{it})\right|dt\right). \tag{39}$$

Corollary 22. *Every function ψ satisfying the conditions of Lemma 21 also satisfies*

$$|\psi(0)| \leq \left(\frac{1}{2\pi(1-\theta)}\int_{2\pi\theta}^{2\pi}\left|\psi(e^{it})\right|dt\right)^{1-\theta}\left(\frac{1}{2\pi\theta}\int_0^{2\pi\theta}\left|\psi(e^{it})\right|dt\right)^{\theta}. \tag{40}$$

Proof of Lemma 21. Let X be a non trivial Banach space. (We can for example suppose that $X = \mathbb{C}$.) Suppose that $X_0 = X_1 = X$ with equality of norms. Then it is a very simple exercise to show that $[X_0, X_1]_\theta = X$ with equality of norms for each $\theta \in (0,1)$. Let a be an element of X with $\|a\|_X = 1$. Then the function $f(z) = \psi(z)a$ is in $\mathcal{F}_{\mathbb{D}}(X_0, X_1) = \mathcal{F}_{\mathbb{D}}(X, X)$. In this case the function h defined by (33) satisfies

$$h(e^{it}) = \left|\psi(e^{it})\right| \text{ for all } t \in [0, 2\pi]\ . \tag{41}$$

for all $t \in [0, 2\pi]$. Furthemor

$$\|f(0)\|_{[X_0,X_1]_\theta} = \|f(0)\|_X = |\psi(0)| \,. \tag{42}$$

To obtain (39), all we have to do now is to substitute (41) and (42) in (36) and let ρ tend to 0. $\square$

Proof of Corollary 22. We write

$$\begin{aligned}
&\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\log\left|\psi(e^{it})\right|dt\right)\\
= &\left[\exp\left(\frac{1}{2\pi}\int_{2\pi\theta}^{2\pi}\log\left|\psi(e^{it})\right|dt\right)\right]\cdot\left[\exp\left(\frac{1}{2\pi}\int_0^{2\pi\theta}\log\left|\psi(e^{it})\right|dt\right)\right]\\
= &\left[\exp\left(\frac{1}{2\pi(1-\theta)}\int_{2\pi\theta}^{2\pi}\log\left|\psi(e^{it})\right|dt\right)\right]^{1-\theta}\cdot\left[\exp\left(\frac{1}{2\pi\theta}\int_0^{2\pi\theta}\log\left|\psi(e^{it})\right|dt\right)\right]^{\theta}
\end{aligned}$$

and then simply apply Jensen's inequality to each of the two factors in the last line. $\square$

Remark 23. *We have already indicated that (39) (and therefore also (40)) also holds for much more general functions ψ. In particular the condition $\psi(1) = \psi(e^{i2\pi\theta}) = 0$ is completely unnecessary and it is almost embarrassing to impose it. However the proof of (39) without imposing this condition would not have the great convenience of using the proof of Lemma 20.*

2.4. An alternative definition of the Calderón "upper" method.

Definition 24. *Let (X_0, X_1) be a regular Banach couple. Let $\mathcal{H}(X_0^\#, X_1^\#)$ be the space of functions $h : \mathbb{S}^\circ \to X_0^\# + X_1^\#$ with the following properties:*

(i) For each fixed $b \in X_0 \cap X_1$, the function $z \mapsto \langle b, h(z)\rangle$ is an element of $H^\infty(\mathbb{S}^\circ)$.

(ii) There exists a constant $C > 0$ such that

$$|\langle b, h(z)\rangle| \le C \|b\|_{X_0}^{1-\operatorname{Re} z} \|b\|_{X_1}^{\operatorname{Re} z} \text{ for all } z \in \mathbb{S}^\circ \text{ and for each } b \in X_0 \cap X_1. \tag{43}$$

For each $h \in \mathcal{H}(X_0^\#, X_1^\#)$, the quantity $\|h\|_{\mathcal{H}(X_0^\#,X_1^\#)}$ is defined by

$$\|h\|_{\mathcal{H}(X_0^\#,X_1^\#)} = \inf C \tag{44}$$

where the infimum is taken over all constants C which satisfy condition (ii).

Remark 25. *Note that (43) implies that, for $j = 0, 1$,*

$$\lim_{r\searrow 0}\sup\left\{|\langle b, h(z)\rangle| : z \in \mathbb{S}^\circ, |\operatorname{Re} z - j| < r\right\} \le C \|b\|_{X_j} .$$

Remark 26. *It might in some sense have seemed more natural in Definition 24 to impose, instead of (ii), a condition on the boundary values of h, i.e., to require that $h(j + it)$ is defined in some perhaps "weak" sense for a.e. t and is an essentially bounded $X_j^\#$ valued function. Indeed, in the more general context of* [4] *(see* [4] *Definition 3.1 pp. 211–212) this is apparently the most natural way to proceed. But for our purposes here it turns out to be technically simpler to impose the (essentially equivalent) condition (ii) and thus to simply bypass the issues of the existence of weak star nontangential limits etc.*

It can be verified that the functional defined by (44) is a norm on the space $\mathcal{H}(X_0^\#, X_1^\#)$ and that $\mathcal{H}(X_0^\#, X_1^\#)$ is a Banach space with respect to this norm. But these facts will also follow immediately from the next theorem.

Theorem 27. *Let (X_0, X_1) be a regular Banach couple.*

(i) Suppose that the function $f : \mathbb{S} \to X_0^\# + X_1^\#$ is an element of $\overline{\mathcal{F}}(X_0^\#, X_1^\#)$. Then its restriction $f|_{\mathbb{S}^\circ}$ to $\mathbb{S}^\circ$ has derivative f' in $\mathcal{H}(X_0^\#, X_1^\#)$ and

$$\|f\|_{\overline{\mathcal{F}}(X_0^\#, X_1^\#)} = \left\|(f|_{\mathbb{S}^\circ})'\right\|_{\mathcal{H}(X_0^\#, X_1^\#)} . \tag{45}$$

(ii) Conversely, suppose that the function $h : \mathbb{S}^\circ \to X_0^\# + X_1^\#$ is an element of $\mathcal{H}(X_0^\#, X_1^\#)$. Then there exists a function $f : \mathbb{S} \to X_0^\# + X_1^\#$ in $\overline{\mathcal{F}}(X_0^\#, X_1^\#)$ whose restriction $f|_{\mathbb{S}^\circ}$ to $\mathbb{S}^\circ$ has its derivative equal to h and

$$\|f\|_{\overline{\mathcal{F}}(X_0^\#, X_1^\#)} = \|h\|_{\mathcal{H}(X_0^\#, X_1^\#)} . \tag{46}$$

An immediate and obvious consequence of this theorem is

Corollary 28. *The space $[X_0^\#, X_1^\#]^\theta$ coincides with the set of all elements $a \in X_0^\# + X_1^\#$ which arise as the values $a = h(\theta)$ at θ of some element $h \in \mathcal{H}(X_0^\#, X_1^\#)$. Furthermore*

$$\|a\|_{[X_0^\#, X_1^\#]^\theta} = \inf\left\{\|h\|_{\mathcal{H}(X_0^\#, X_1^\#)} : h \in \mathcal{H}(X_0^\#, X_1^\#), h(\theta) = a\right\}.$$

Proof of Theorem 27. First, for part (i), let us suppose that $f \in \overline{\mathcal{F}}(X_0^\#, X_1^\#)$ with $\|f\|_{\overline{\mathcal{F}}(X_0^\#, X_1^\#)} = 1$. We consider the functions $f_n : \mathbb{S} \to X_0^\# + X_1^\#$ defined for each $n \in \mathbb{N}$ by

$$f_n(z) = \frac{n}{i}\left(f(z + i/n) - f(z)\right).$$

Each function f_n is of course a continuous map from $\mathbb{S}$ into $X_0^\# + X_1^\#$. We do not know a priori that f_n is bounded. However, because of the definition of the space $\overline{\mathcal{F}}(X_0^\#, X_1^\#)$, we know that

$$\|f_n(z)\|_{X_0^\# + X_1^\#} \le c(1 + |z|) \text{ for all } z \in \mathbb{S} \tag{47}$$

for some constant c which may depend on n. We also have $f_n(j + it) \in X_j^\#$ with

$$\|f_n(j + it)\|_{X_j^\#} \le 1 \text{ for } j = 0, 1 \text{ and all } t \in \mathbb{R}.$$

For each fixed $b \in X_0 \cap X_1$ we consider the continuous scalar function $\phi_{n,b} : \mathbb{S} \to \mathbb{C}$ defined by $\phi_{n,b}(z) = \langle b, f_n(z)\rangle$. This is clearly analytic in $\mathbb{S}^\circ$ and satisfies

$$|\phi_{n,b}(j + it)| \le \|b\|_{X_j} \le \|b\|_{X_0 \cap X_1} \text{ for } j = 0, 1 \text{ and all } t \in \mathbb{R}. \tag{48}$$

and

$$|\phi_{n,b}(z)| \le c\,\|b\|_{X_0 \cap X_1}(1 + |z|) \text{ for all } z \in \mathbb{S}.$$

So we can apply Lemma 11 to obtain that

$$|\phi_{n,b}(z)| \le \|b\|_{X_0}^{1-\operatorname{Re} z}\|b\|_{X_1}^{\operatorname{Re} z} \le \|b\|_{X_0 \cap X_1} \text{ for all } z \in \mathbb{S}^\circ. \tag{49}$$

Passing to the limit as n tends to ∞, we see that the $X_0^\# + X_1^\#$ valued analytic function $h := (f|_{\mathbb{S}^\circ})'$ satisfies

$$|\langle b, h(z)\rangle| \le \|b\|_{X_0}^{1-\operatorname{Re} z}\|b\|_{X_1}^{\operatorname{Re} z} \le \|b\|_{X_0 \cap X_1} \text{ for all } z \in \mathbb{S}^\circ. \tag{50}$$

So h satisfies conditions (i) and (ii) of Definition 24 with $C = 1$. This establishes part (i) of Theorem 27, except that at this stage, instead of (45), we only have

$$\|f\|_{\overline{\mathcal{F}}(X_0^\#,X_1^\#)} \geq \left\|(f|_{\mathbb{S}^\circ})'\right\|_{\mathcal{H}(X_0^\#,X_1^\#)}. \tag{51}$$

Now, turning to part (ii) of the theorem, we suppose that $h : \mathbb{S} \to X_0^\# + X_1^\#$ is an element of $\mathcal{H}(X_0^\#, X_1^\#)$ with norm 1. Then, for each fixed $b \in X_0 \cap X_1$, the function $h_b : \mathbb{S}^\circ \to \mathbb{C}$ defined by $h_b(z) = \langle b, h(z)\rangle$ is an element of $H^\infty(\mathbb{S}^\circ)$. It follows immediately from condition (43) that

$$|h_b(z)| \leq \|b\|_{X_0}^{1-\operatorname{Re} z} \|b\|_{X_1}^{\operatorname{Re} z} \leq \|b\|_{X_0\cap X_1}^{1-\operatorname{Re} z} \|b\|_{X_0\cap X_1}^{\operatorname{Re} z} = \|b\|_{X_0\cap X_1} \text{ for all } z \in \mathbb{S}^\circ. \tag{52}$$

Let $\phi_b : \mathbb{S} \to \mathbb{C}$ be the function obtained from h_b as in Lemma 15, by setting $\phi_b(z) = \int_{1/2}^z h_b(\zeta)d\zeta$ for all $z \in \mathbb{S}^\circ$ and then extending continuously to $\mathbb{S}$. In view of condition (iv) of Lemma 15 and (52) we have

$$|\phi_b(z)| \leq \left|z - \frac{1}{2}\right| \cdot \|b\|_{X_0\cap X_1} \text{ for all } z \in \mathbb{S}. \tag{53}$$

Since $h_b(z)$ depends linearly on b at each constant point $z \in \mathbb{S}^\circ$, it follows that $\phi_b(z)$ also depends linearly on b at each constant point $z \in \mathbb{S}$. In view of this and (53), for each $z \in \mathbb{S}$ there exists a unique element $f(z) \in (X_0 \cap X_1)^\# = X_0^\# + X_1^\#$ such that

$$\langle b, f(z)\rangle = \phi_b(z) \text{ for all } b \in X_0 \cap X_1 \tag{54}$$

and

$$\|f(z)\|_{X_0^\#+X_1^\#} \leq \left|z - \frac{1}{2}\right|. \tag{55}$$

Condition (i) of Lemma 15 gives us that $|\phi_b(z_1) - \phi_b(z_2)| \leq |z_1 - z_2| \cdot \|b\|_{X_0\cap X_1}$ for all $b \in X_0 \cap X_1$ and consequently $\|f(z_1) - f(z_2)\|_{X_0^\#+X_1^\#} \leq |z_1 - z_2|$ for all $z_1, z_2 \in \mathbb{S}$. This shows that $f : \mathbb{S} \to X_0^\# + X_1^\#$ is continuous.

From (54) and the fact that $X_0 \cap X_1$ is the predual of $X_0^\# + X_1^\#$ and some standard results which are recalled in Appendix 3.1 we see that f is an analytic $X_0^\# + X_1^\#$ valued function on $\mathbb{S}^\circ$. Furthermore, it follows from (54) that (cf. condition (ii) of Lemma 15) we must have $f'(z) = h(z)$ for each $z \in \mathbb{S}^\circ$.

Condition (55) and the discussion following it show that f satisfies the first three of the four conditions for membership in the space $\overline{\mathcal{F}}(X_0^\#, X_1^\#)$ listed in Section 5 on p. 115 of [3]. To obtain the fourth condition we first deduce from (52) that (cf. Remark 25) $\lim_{r\searrow 0} (\sup\{|h_b(z)| : |\operatorname{Re} z - j| < r\}) \leq \|b\|_{X_j}$ for $j = 0, 1$. Then condition (iii) of Lemma 15 applied to $h_b(z)$ tells us that

$$|\phi_b(j + it_1) - \phi_b(j + it_2)| \leq |t_1 - t_2| \cdot \|b\|_{X_j}$$

for $j = 0, 1$ and all $t_1, t_2 \in \mathbb{R}$ and all $b \in X_0 \cap X_1$.

In view of this and (54) it follows that

$$\|f\|_{\overline{\mathcal{F}}(X_0^\#,X_1^\#)} = \sup\left\{\frac{\|f(j + it_1) - f(j + it_2)\|_{X_j^\#}}{|t_1 - t_2|} : j = 0, 1, t_1, t_2 \in \mathbb{R}, t_1 \neq t_2\right\} \leq 1.$$

This establishes the above–mentioned fourth condition, i.e., that $f \in \overline{\mathcal{F}}(X_0^\#, X_1^\#)$, and it also shows that in general

$$\|f\|_{\overline{\mathcal{F}}(X_0^\#, X_1^\#)} \leq \|h\|_{\mathcal{H}(X_0^\#, X_1^\#)} . \tag{56}$$

But now, since $f' = h$ on $\mathbb{S}^\circ$, we can also apply the first part of the proof to obtain the reverse inequality to (56), i.e., (51). This gives us (46). Analogously, in the context of the first part of this proof, we can now obtain the reverse inequality to (51) for any given $f \in \overline{\mathcal{F}}(X_0^\#, X_1^\#)$, by applying the second part of the proof and (56) to the function $h \in \mathcal{H}(X_0^\#, X_1^\#)$ defined by $h(z) = h := (f|_{\mathbb{S}^\circ})'$. This establishes (45) and so completes the proof of Theorem 27. $\square$

2.5. **Our proof of Theorem 10.**

Part 1: The inclusion $[X_0^\#, X_1^\#]^\theta \overset{1}{\subset} ([X_0, X_1]_\theta)^\#$.

Suppose first that y is an element of $[X_0^\#, X_1^\#]^\theta$ and that ϵ is an arbitrary positive number. By Theorem 27 and its corollary, there exists a continuous function $h \in \mathcal{H}(X_0^\#, X_1^\#)$ such that $\|h\|_{\mathcal{H}(X_0^\#, X_1^\#)} \leq (1+\epsilon) \|y\|_{[X_0^\#, X_1^\#]^\theta}$ and $h(\theta) = y$.

Now suppose that x is an arbitrary element of $X_0 \cap X_1$ with norm $\|x\|_{[X_0,X_1]_\theta} < 1$. In view of Stafney's theorem ([15] Lemma 2.5 p. 335) there exists a function $g \in \mathcal{G}(X_0, X_1)$ such that $g(\theta) = x$ and

$$\sup_{t\in\mathbb{R}} \|g(j+it)\|_{X_j} < 1 \text{ for } j = 0, 1. \tag{57}$$

We recall that membership in $\mathcal{G}(X_0, X_1)$ means that $g : \mathbb{S} \to X_0 \cap X_1$ is a finite sum of the form

$$g(z) = \sum_{n=1}^{N} \phi_n(z) a_n \tag{58}$$

where each $a_n \in X_0 \cap X_1$ and each $\phi_n : \mathbb{C} \to \mathbb{C}$ is an entire function (of a special form) which is bounded on $\mathbb{S}$ and satisfies $\lim_{R\to\infty} \sup \{|\phi_n(z)| : z \in \mathbb{S}, |\mathrm{Im}\, z| \geq R\} = 0$ It is clear that, for $j = 0, 1$, the function $z \mapsto \|g(z)\|_{X_j}$ is a bounded and uniformly continuous function on $\mathbb{S}$. Its uniform continuity, together with (57) implies that, for some sufficiently small positive number r,

$$\sup\left\{\|g(z)\|_{X_j} : z \in \mathbb{S}, 0 < |j - \mathrm{Re}\ z| < r\right\} < 1. \tag{59}$$

We now consider the function $\Phi : \mathbb{S}^\circ \to \mathbb{C}$ defined by

$$\Phi(z) = \langle g(z), h(z)\rangle = \sum_{n=1}^{N} \phi_n(z) \langle a_n, h(z)\rangle .$$

This is a finite sum of products of functions in $H^\infty(\mathbb{S}^\circ)$ and is therefore itself in $H^\infty(\mathbb{S}^\circ)$. In view of (43) and (44) in Definition 24, we also have

$$|\Phi(z)| \leq (1+\epsilon) \|y\|_{[X_0^\#, X_1^\#]^\theta} \|g(z)\|_{X_0}^{1-\mathrm{Re}\, z} \|g(z)\|_{X_1}^{\mathrm{Re}\, z} \text{ for all } z \in \mathbb{S}^\circ. \tag{60}$$

Using (60), (59) and the boundedness on $\mathbb{S}$ of the functions $z \mapsto \|g(z)\|_{X_j}$, we obtain that

$$\lim_{r\searrow 0} \sup \{|\Phi(z)| : z \in \mathbb{S},\ 0 < |j - \mathrm{Re}\, z| < r\} \leq (1+\epsilon) \|y\|_{[X_0^\#, X_1^\#]^\theta} \text{ for } j = 0, 1.$$

Thus, by part (ii) of Lemma 11, it follows that

$$|\langle x, y\rangle| = |\langle g(\theta), h(\theta)\rangle| = |\Phi(\theta)| \leq (1+\epsilon)\|y\|_{[X_0^{\#}, X_1^{\#}]^{\theta}}.$$

Since x is an arbitrary element of $X_0 \cap X_1$, and since ϵ can be chosen arbitrarily small, it follows that $y \in ([X_0, X_1]_\theta)^{\#}$ with $\|y\|_{([X_0,X_1]_\theta)^{\#}} \leq \|y\|_{[X_0^{\#}, X_1^{\#}]^{\theta}}$.

Part 2: The inclusion $([X_0, X_1]_\theta)^{\#} \overset{1}{\subset} [X_0^{\#}, X_1^{\#}]^{\theta}$.

Let us now suppose that y is an arbitrary element of $([X_0, X_1]_\theta)^{\#}$. We shall use y to define a linear functional λ on the space $\mathcal{G}_{\mathbb{D}}(X_0, X_1)$ (see Definition 17) by setting $\lambda(f) = \langle f(0), y\rangle$ for each $f \in \mathcal{G}_{\mathbb{D}}(X_0, X_1)$.

(Note that $f(0) \in X_0 \cap X_1$ whenever $f \in \mathcal{G}_{\mathbb{D}}(X_0, X_1)$, and, (cf. Fact 3 and (1)) we also have $([X_0, X_1]_\theta)^{\#} \subset (X_0 \cap X_1)^{\#}$ so the notation $\langle f(0), y\rangle$ is appropriate.)

Instead of using the norm (28), we shall equip $\mathcal{G}_{\mathbb{D}}(X_0, X_1)$ with the norm

$$\|f\|_L = \frac{1}{2\pi}\left(\int_{2\pi\theta}^{2\pi} \left\|f(e^{it})\right\|_{X_0} dt + \int_0^{2\pi\theta} \left\|f(e^{it})\right\|_{X_1} dt\right) \tag{61}$$

In view of (32), we see that λ is bounded on $(\mathcal{G}_{\mathbb{D}}(X_0, X_1), \|\cdot\|_L)$ with norm not exceeding $\|y\|_{([X_0,X_1]_\theta)^{\#}}$.

Let L be the space of (equivalence classes of) functions $f : \mathbb{T} \to X_0 \cap X_1$ which are finite sums of the form $f(w) = \sum_{k=1}^{N} \psi_k(w) a_k$, where, again, as in Definition 17, each a_k is in $X_0 \cap X_1$, but now each $\psi_k : \mathbb{T} \to \mathbb{C}$ is only defined on $\mathbb{T}$, and $\psi(e^{it})$ is an integrable function of t on $[0, 2\pi]$. We use the same norm (61) for L as we did for $\mathcal{G}_{\mathbb{D}}(X_0, X_1)$. The fact that each function $f \in L$ takes values in a finite dimensional subspace of $X_0 \cap X_1$ ensures that $t \mapsto \left\|f(e^{it})\right\|_{X_j}$ is Lebesgue measurable for $j = 0, 1$ and thus the two integrals in (61) are well defined.

The Hahn–Banach theorem guarantees the existence of a bounded linear functional $\widetilde{\lambda} : L \to \mathbb{C}$ which is an extension of λ and which satisfies $\left|\widetilde{\lambda}(f)\right| \leq \|y\|_{([X_0,X_1]_\theta)^{\#}} \cdot \|f\|_L$ for each L.

Let $L^1(\mathbb{T})$ denote the space of all (equivalence classes of) measurable functions $\phi : \mathbb{T} \to \mathbb{C}$ for which $\|\phi\|_{L^1(\mathbb{T})} := \frac{1}{2\pi}\int_0^{2\pi} \left|\phi(e^{it})\right| dt < \infty$. For each fixed $b \in X_0 \cap X_1$, we define a linear functional $\widetilde{\lambda}_b$ on this space by the formula

$$\widetilde{\lambda}_b(\phi) = \widetilde{\lambda}(\phi b) \tag{62}$$

for all $\phi \in L^1(\mathbb{T})$. Here of course ϕb means the function in L defined by $(\phi b)(e^{it}) = \phi(e^{it}) b$. Thus, for each $\phi \in L^1(\mathbb{T})$, we have

$$\begin{aligned}
\left|\widetilde{\lambda}_b(\phi)\right| &\leq \|y\|_{([X_0,X_1]_\theta)^{\#}} \cdot \|\phi b\|_L \\
&= \|y\|_{([X_0,X_1]_\theta)^{\#}} \cdot \frac{1}{2\pi}\left(\|b\|_{X_0}\int_{2\pi\theta}^{2\pi} \left|\phi(e^{it})\right| dt + \|b\|_{X_1}\int_0^{2\pi\theta} \left|\phi(e^{it})\right| dt\right).
\end{aligned}$$

By standard results, we see that there exists an essentially bounded measurable function $h_b : \mathbb{T} \to \mathbb{C}$ such that

$$\widetilde{\lambda}_b(\phi) = \frac{1}{2\pi}\int_0^{2\pi} \phi(e^{it}) h_b(e^{it}) dt \text{ for all } \phi \in L^1(\mathbb{T}) \tag{63}$$

and

$$\operatorname*{ess\,sup}_{e^{it}\in\Gamma_j}\left|h_b(e^{it})\right| \leq \|y\|_{([X_0,X_1]_\theta)^\#}\cdot\|b\|_{X_j} \text{ for } j=0,1. \tag{64}$$

For each positive constant $\delta>0$, we define the function $\Omega_\delta:\mathbb{D}\to\mathbb{C}$ by

$$\Omega_\delta(w):=\begin{cases} e^{\delta(\xi(w))^2} & , \quad w\in\mathbb{D}\backslash\{1,e^{i2\pi\theta}\} \\ 0 & , \quad w=1,e^{i2\pi\theta}\end{cases}.$$

This is a special case of the function f, initially defined by $f=\phi\circ\xi$, used in the proof of part (vi) of Theorem 13. (Just set $M_0=M_1=1$ there.) The explanations given in that previous proof show that Ω_δ is continuous on $\mathbb{D}$ and analytic in $\mathbb{D}^\circ$. Consequently, for each $k\in\mathbb{N}$ and each $b\in X_0\cap X_1$, the function $f_{\delta,k,b}(w):=w^k\Omega_\delta(w)b$ is an element of $\mathcal{G}_{\mathbb{D}}(X_0,X_1)$. So we have

$$\begin{aligned} 0 &= \langle f_{\delta,k,b}(0),y\rangle=\lambda(f_{\delta,k,b})=\widetilde{\lambda}(f_{\delta,k,b})=\widetilde{\lambda}_b(w^k\Omega_\delta(w)) \\ &= \frac{1}{2\pi}\int_0^{2\pi}e^{ikt}e^{\delta\left(\xi(e^{it})\right)^2}h_b(e^{it})dt. \end{aligned}$$

By taking the limit as δ tends to 0, and applying the dominated convergence theorem, we obtain that $\frac{1}{2\pi}\int_0^{2\pi}e^{ikt}h_b(e^{it})dt=0$ for all $k\in\mathbb{N}$. I.e., the function h_b satisfies the condition in part (iii) of Theorem 13, and obviously also (17). Therefore the function $u_b:\mathbb{D}^\circ\to\mathbb{C}$ defined for each $w\in\mathbb{D}^\circ$ by the absolutely convergent series (cf. (18))

$$u_b(w):=\sum_{n=1}^\infty\left(\frac{1}{2\pi}\int_0^{2\pi}e^{-ins}h_b(e^{is})ds\right)w^n$$

is analytic in $\mathbb{D}^\circ$. Part (iv) of Theorem 13, together with (64) gives us that

$$|u_b(w)|\leq\|y\|_{([X_0,X_1]_\theta)^\#}\cdot\|b\|_{X_0\cap X_1} \text{ for each } w\in\mathbb{D}^\circ. \tag{65}$$

Part (vi) of Theorem 13, together with (64) gives us that the bounded analytic function $\phi_b:\mathbb{S}^\circ\to\mathbb{C}$ defined by $\phi_b(z)=u_b(\xi^{-1}(z))$ satisfies

$$|\phi_b(z)|=\left|u_b(\xi^{-1}(z))\right|\leq\|y\|_{([X_0,X_1]_\theta)^\#}\cdot\|b\|_{X_0}^{1-\operatorname{Re}z}\cdot\|b\|_{X_1}^{\operatorname{Re}z} \text{ for all } z\in\mathbb{S}^\circ. \tag{66}$$

Each Fourier coefficient $\frac{1}{2\pi}\int_0^{2\pi}e^{-ins}h_b(e^{is})ds$ depends linearly on b since, when we apply (63) and then (62) to the function $\phi:\mathbb{T}\to\mathbb{C}$ defined by $\phi(e^{it})=e^{-nt}$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}e^{-ins}h_b(e^{is})ds=\widetilde{\lambda}_b(\phi)=\widetilde{\lambda}(\phi b).$$

It follows that $u_b(w)$ and $\phi_b(z)$ also depend linearly on b, for each fixed $w\in\mathbb{D}^\circ$ and each fixed $z\in\mathbb{S}^\circ$. Thus, by (65) these are both bounded linear functionals on $X_0\cap X_1$, and, for each $z\in\mathbb{S}^\circ$, there exists an element $v(z)\in(X_0\cap X_1)^\#=X_0^\#+X_1^\#$ such that $\langle b,v(z)\rangle=\phi_b(z)$. The properties of ϕ_b stated above, notably (66), imply that $v\in\mathcal{H}(X_0^\#,X_1^\#)$ with $\|v\|_{\mathcal{H}(X_0^\#,X_1^\#)}\leq\|y\|_{([X_0,X_1]_\theta)^\#}$. Thus it remains only to show that $v(\theta)=y$, i.e., that $\langle b,v(\theta)\rangle=\langle b,y\rangle$ for each $b\in X_0\cap X_1$. In view of the various definitions given in the preceding steps, we see that

$$\begin{aligned}
\langle b, v(\theta)\rangle &= \phi_b(\theta) = u_b(\xi^{-1}(\theta)) = u_b(0) = \frac{1}{2\pi}\int_0^{2\pi} h_b(e^{is})ds \\
&= \lim_{\delta\to 0}\frac{1}{2\pi}\int_0^{2\pi} e^{\delta\xi(e^{it})^2} h_b(e^{is})ds \\
&= \lim_{\delta\to 0}\widetilde{\lambda}_b\left(\Omega_\delta\right) = \lim_{\delta\to 0}\widetilde{\lambda}\left(\Omega_\delta b\right) = \lim_{\delta\to 0}\lambda\left(\Omega_\delta b\right) \\
&= \lim_{\delta\to 0}\langle\Omega_\delta(0)b, y\rangle = \langle b, y\rangle \ ,
\end{aligned}$$

which indeed completes the proof. □

3. Appendices

Well, so far, there is only one appendix.

3.1. **Banach space valued analytic functions.** Let us here recall some basic definitions and properties of analytic functions taking values in a Banach space. Our reference will be the old treatise [7] of Hille and Phillips.

Let A be a Banach space, let $D \subset \mathbb{C}$ be a domain. We recall (cf. [7] Definition 3.10.1, pp. 92–93) that a A–valued function $f : D \to A$ is said to be analytic (or holomorphic) in D if the scalar function $z \mapsto y(f(z))$ is analytic in D for each choice of the constant linear functional y in some *"determining manifold"* Y in A^*, i.e., (cf. [7] Definition 2.8.6, p. 34) some closed subspace Y of A^* such that

$$\|a\|_A = \sup\left\{|y(a)| : y \in Y, \|y\|_{A^*} \leq 1\right\} \text{ for all } a \in A. \tag{67}$$

In particular, if A has a predual which is identified isometrically with the space Y in the usual canonical way, then of course this condition is fulfilled. (This happens, e.g., in the second part of the proof of Theorem 27 above in Section 2, where we have $Y = X_0 \cap X_1$ and $A = X_0^{\#} + X_1^{\#} = (X_0 \cap X_1)^{\#}$.)

It follows from arguments using the uniform bounded principle ([7] pp. 93–97) that, perhaps surprisingly, this seemingly rather weak condition is equivalent to a seemingly much stronger condition, namely that, for each open disc $\{z : |z - z_0| < R\}$ contained in D there exists a sequence of elements $\{a_n\}_{n\geq 0}$ such that the series $\sum_{n=0}^{\infty}(z - z_0)^n a_n$ converges absolutely in A norm to $f(z)$ at every point of the disc. (Of course the elements a_n have to satisfy $a_n = \frac{1}{n!}f^{(n)}(z_0)$ where the nth order derivative $f^{(n)}$ of f can be defined in several different but of course equivalent ways.)

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